A Perturbation Procedure for Nearly Rectangular, Homogeneously Filled, Cylindrical Waveguides

Brian J. McCartin

Abstract—**The cut-off frequencies and propagating modes of a hollow cylindrical waveguide may be approximated by conformal mapping to a canonical domain followed by the numerical solution of a Helmholtz-like equation [1]. This letter considers the problem of how these frequencies and modes change under a small perturbation of the bounding metallic walls. A procedure is herein presented that produces a perturbation expansion involving only computations in the unperturbed cross section, thus avoiding costly additional mappings. Moreover, the resulting analytical expressions for these frequencies and modes are then available for optimization of waveguide parameters. An application of this procedure is presented together with comparison to published numerical and experimental results.**

I. Introduction

A METHOD FOR calculating the modes of waveguides of very general cross section by conformal transformation to a rectangle has been presented in [1]. The general problem of the conformal mapping of a "towel-shaped" region, such as a near-rectangle, onto a rectangle has been treated in [2]. In this letter, we combine these analyzes with a procedure due to Schrödinger [3] to produce a perturbation expansion for the cut-off frequencies and modal shapes of a nearly rectangular, homogeneously filled, cylindrical waveguide.

The procedure presented herein requires only a single conformal mapping yet produces an analysis for an entire one-parameter family of related waveguides. Hence, the resulting analytical expressions for cutoff frequencies and modal shapes are available for the optimal design of waveguide parameters.

All numerical computations were performed using MATLAB©. An application of this perturbation procedure to a high-power waveguide is presented. The cutoff wavelengths so computed are compared to published numerical and experimental results.

II. Formulation

Consider a nearly rectangular waveguide cross section such as that appearing in Fig. 1. It is assumed that the waveguide is uniform in the longitudinal direction and is homogeneously filled with a lossless, isotropic material characterized by permittivity, ϵ, and permeability, μ. Such a nearly rectangular geometry is "towel-shaped," and as a consequence may be mapped conformally onto a rectangle by the technique of [2]. The aspect ratio of the rectangle (conformal module) may not be specified but must be determined. The rectangle is scaled to have the same area as the physical waveguide cross section.

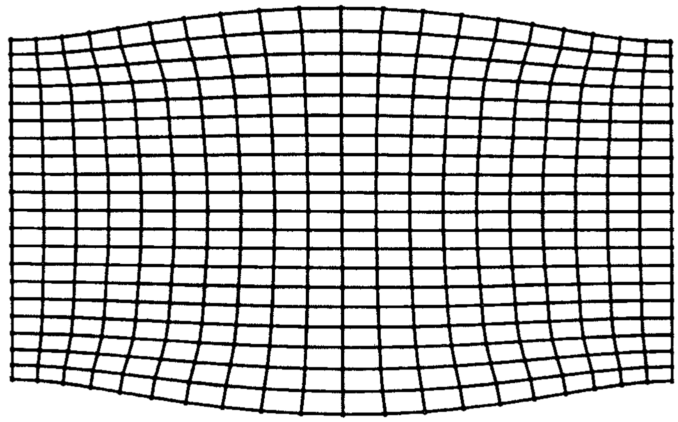

Fig. 1. Waveguide cross section.

Denoting the physical plane as the $w(= u + \imath v)$ plane and the mapped plane as the $z(= x + \imath y)$ plane, the Helmholtz equation for the waveguide modes maps to [4]

$$\phi_{xx} + \phi_{yy} + (k^2 - \beta^2)|f'(z)|^2\phi = 0 \qquad (1)$$

under the conformal transformation $w = f(z)$, where $k^2 := \omega^2\mu\epsilon$ and β is the propagation constant. Along the metallic walls, $\phi = 0$ for TM-modes and $\phi_n = 0$ for TE-modes.

Once such a mapping function has been constructed, we may consider the one-parameter family of related conformal mappings

$$w = f_\delta(z) := z + \delta(f(z) - z) \qquad (2)$$

yielding the area correspondence

$$|f_\delta'|^2 = 1 + \delta(2(\Re e f' - 1)) + \delta^2(1 - 2\Re e f' + |f'|^2). \qquad (3)$$

If the original waveguide was of nearly rectangular cross section, then $f(z) \approx z$ and $|f_\delta'(z)|^2 \approx 1$ for $\delta = O(1)$.

The transformed Helmholtz equation

$$\phi_{xx} + \phi_{yy} = \lambda(1 + \delta a(x, y) + \delta^2 b(x, y))\phi \qquad (4)$$

where $\lambda := \beta^2 - k^2$ and a and b are defined implicitly by (3), must now be satisfied in the interior of the rectangle subject to either Dirichlet (TM) or Neumann (TE) boundary conditions. Note the correspondences $\delta = 0$ to the rectangle and $\delta = 1$ to the original waveguide.

If we discretize this problem using finite differences [5] with constant spatial increments, $(\Delta x, \Delta y)$, we obtain the matrix

Manuscript received April 4, 1996.
The author is with the GMI Engineering and Management Institute, Flint, MI 48504-4998 USA.
Publisher Item Identifier S 1051-8207(96)07485-5.

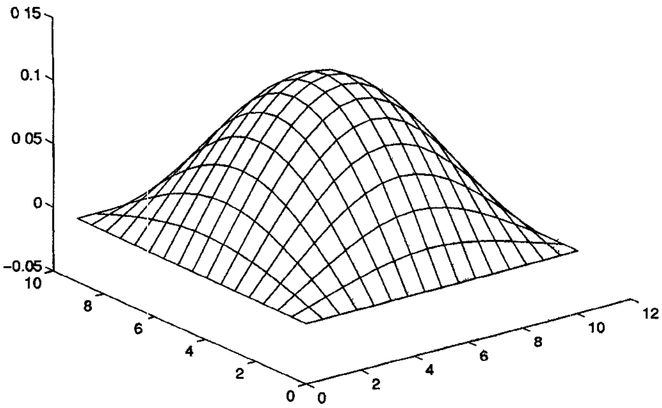

Fig. 2. $TM_{1,1}$ Mode.

generalized eigenvalue problem

$$M\phi = \lambda(N + \delta A + \delta^2 B)\phi \quad (5)$$

where M is symmetric and nonpositive definite, N is positive and diagonal, and both A and B are diagonal. The cutoff wavelength of a mode is related to the corresponding generalized eigenvalue by $\lambda_c = 2\pi/\sqrt{|\lambda|}$.

III. Perturbation Procedure

We now apply the perturbation procedure of Schrödinger [3] to construct expansions

$$\lambda(\delta) = \sum_{n=0}^{\infty} \delta^n \lambda_n; \quad \phi(\delta) = \sum_{n=0}^{\infty} \delta^n \phi_n \quad (6)$$

the convergence of which are studied in [6].

Here, we assume that λ_0 is a simple eigenvalue with corresponding eigenvector ϕ_0 for the unperturbed problem

$$M\phi_0 = \lambda_0 N \phi_0 \quad (7)$$

which simply involves the discrete Laplacian whose eigenvalues and eigenvectors on a rectangle are well known [5]. An eigenvalue of multiplicity m would entail an expansion in $\delta^{1/m}$ [7]. As will be seen in what follows, our approximation scheme will reap big dividends from the symmetry of M [8].

Inserting the expansions (6) into the eigenproblem (5), collecting terms, and equating the coefficient of each power of δ to zero results in

$$\begin{aligned}(M - \lambda_0 N)\phi_0 &= 0 && (8)\\ (M - \lambda_0 N)\phi_1 &= (\lambda_1 N + \lambda_0 A)\phi_0 && (9)\\ (M - \lambda_0 N)\phi_2 &= (\lambda_1 N + \lambda_0 A)\phi_1 \\ &\quad + (\lambda_2 N + \lambda_1 A + \lambda_0 B)\phi_0 && (10)\\ (M - \lambda_0 N)\phi_3 &= (\lambda_1 N + \lambda_0 A)\phi_2 \\ &\quad + (\lambda_2 N + \lambda_1 A + \lambda_0 B)\phi_1 \\ &\quad + (\lambda_3 N + \lambda_2 A + \lambda_1 B)\phi_0 && (11)\end{aligned}$$

and so forth. Equation (8) together with the normalization $\langle\phi_0, N\phi_0\rangle = 1$ yields $\lambda_0 = \langle\phi_0, M\phi_0\rangle$. Note that $(M - \lambda_0 N)$ is singular (in fact, its nullity is one by assumption) and that the symmetry of M implies that the right-hand side of (9) must be orthogonal to ϕ_0 producing

$$\lambda_1 = -\lambda_0\langle\phi_0, A\phi_0\rangle. \quad (12)$$

Thus, the symmetry of M has produced λ_1 *without* calculating ϕ_1. Since A is indefinite, λ_1 can be either positive or negative. Consequently, the perturbation approximation provides neither lower nor upper bounds for the eigenvalues.

In order to proceed any further, however, we must calculate ϕ_1 from (9) by performing the QR factorization of $(M - \lambda_0 N)$ and then employ it to produce the minimum-norm least-squares solution, ψ, to this rank-deficient system [9]. We then define $\phi_1 := \psi - \langle\psi, N\phi_0\rangle\phi_0$ producing $\langle\phi_1, N\phi_0\rangle = 0$ which simplifies subsequent computations.

The knowledge of ϕ_1, together with the symmetry of M, permits the computation of *both* λ_2 and λ_3:

$$\begin{aligned}\lambda_2 &= -\lambda_1\langle\phi_0, A\phi_0\rangle - \lambda_0\langle\phi_0, A\phi_1 + B\phi_0\rangle && (13)\\ \lambda_3 &= -\lambda_2\langle\phi_0, A\phi_0\rangle - \lambda_1(\langle\phi_0, 2A\phi_1 + B\phi_0\rangle + \langle\phi_1, N\phi_1\rangle) \\ &\quad - \lambda_0\langle\phi_1, A\phi_1 + 2B\phi_0\rangle. && (14)\end{aligned}$$

In evaluating the required inner products, we benefit greatly from the diagonality of N, A, and B.

We can continue indefinitely in this fashion, with each succeeding term in the expansion for ϕ producing the next two terms in the expansion for λ. If we had reduced this to a standard eigenvalue problem through multiplication of (5) by N^{-1}, we would have destroyed the symmetry of the operator and sacrificed this substantial economy of computation. Also, note that only a single QR factorization is required to produce *all* of the modal corrections, ϕ_n.

IV. Example

We now apply the above perturbation procedure to an analysis of the modal characteristics of the high-power waveguide cross section shown in Fig. 1. This waveguide has width 100 mm and height 82 mm on the sides and 98 mm at the center. Both numerically computed and experimentally measured cutoff wavelengths for various modes of this structure are provided in [1].

Fig. 2 displays our first-order approximation to the $TM_{1,1}$ mode. Table I displays our third-order approximation to the cutoff wavelengths (in cm). In this Table, a superscript of p, n, and e denotes a perturbation, numerical, and experimental value, respectively.

TABLE I
Calculated Cutoff Wavelengths Compared to [1]

Mode	λ_c^p	λ_c^n	λ_c^e
$TE_{1,0}$	19.21	18.90	18.69
$TE_{0,1}$	18.28	18.37	18.30
$TE_{1,1}$	12.64	12.57	12.60
$TM_{1,1}$	13.69	13.63	13.60

In all of our computations, which were performed using MATLAB©, a coarse 9×9 mesh and a fine 17×17 mesh were employed. These results were then enhanced using Richardson extrapolation [10]. One observes a generally close agreement among the results of these three disparate techniques. The magnitude of the discrepancy for the $TE_{1,0}$ mode is unexpected and does not vanish with mesh refinement. In another study [11], a similar discrepancy was observed between an "exact" finite difference procedure and the results of [1] for the fundamental mode. Given the consistency of these discrepancies, one is lead to question the accuracy of the measured results for the fundamental mode. It cannot be too strongly emphasized, however, that only the perturbation procedure developed in this note yields an analytical expression capable of further exploitation in the engineering design process.

V. CONCLUSION

In the preceding sections, we have presented a perturbation procedure for the modal characteristics of nearly rectangular, homogeneously filled, cylindrical waveguides. This procedure has been validated against published numerical and experimental data. The resulting closed-form expressions are then available for waveguide optimization studies.

Although the focus of this note has been on nearly rectangular waveguide cross sections, the procedure described herein is also applicable to small perturbations of other canonical domains, such as circular cross sections, for which the modes are known [4] and numerical procedures for the required conformal mapping are available [12].

REFERENCES

[1] H. H. Meinke, K. P. Lange, and J. F. Ruger, "TE- and TM-waves in waveguides of very general cross section," *Proc. IEEE*, vol. 51, pp. 1436–1443, Nov. 1963.

[2] A. Seidl and H. Klose, "Numerical conformal mapping of a towel-shaped region onto a rectangle," *SIAM J. Sci. Stat. Comput.*, vol. 6, no. 4, pp. 833–842, 1985.

[3] F. W. Byron, Jr. and R. W. Fuller, *Mathematics of Classical and Quantum Physics*. New York: Dover, 1992.

[4] D. S. Jones, *Methods in Electromagnetic Wave Propagation*. New York: Oxford, 1979.

[5] F. B. Hildebrand, *Finite-Difference Equations and Simulations*. Englewood Cliffs, NJ: Prentice-Hall, 1968.

[6] K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space*. American Mathematical Society, 1965.

[7] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*. New York: Oxford, 1965.

[8] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed. New York: Springer-Verlag, 1980.

[9] B. Noble and J. W. Daniel, *Applied Linear Algebra*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1988.

[10] S. D. Conte and C. de Boor, *Elementary Numerical Analysis*, 3rd ed. New York: McGraw-Hill, 1980.

[11] B. J. McCartin, "Numerical Computation of Guided Electromagnetic Waves," in *Proc. 12th Annu. Conf. Applied Mathematics*, Univ. Central Oklahoma, 1996.

[12] H. Kober, *Dictionary of Conformal Representations*. New York: Dover, 1957.